KADEC-PEŁCZYŃSKI DECOMPOSITION FOR HAAGERUP L^p-SPACES

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Abstract. Let $\mathcal{M}$ be a von Neumann algebra (not necessarily semi-finite). We provide a generalization of the classical Kadec-Pełczynski subsequence decomposition of bounded sequences in $L^p[0,1]$ to the case of the Haagerup L^p-spaces $(1 \leq p < \infty)$. In particular, we prove that if $(\varphi_n)_n$ is a bounded sequence in the predual $\mathcal{M}_*$ of $\mathcal{M}$, then there exist a subsequence $(\varphi_{n_k})_k$ of $(\varphi_n)_n$, a decomposition $\varphi_{n_k} = y_k + z_k$ such that $\{y_k,\ k \geq 1\}$ is relatively weakly compact and the support projections $s(z_k) \downarrow_k 0$ (or similarly mutually disjoint). As an application, we prove that every non-reflexive subspace of the dual of any given C^*-algebra (or Jordan triples) contains asymptotically isometric copies of ℓ^1 and therefore fails the fixed point property for nonexpansive mappings. These generalize earlier results for the case of preduals of semi-finite von Neumann algebras.

1. Introduction

In [19], Kadec and Pełczyński proved a fundamental property that if $1 \leq p < \infty$ then every bounded sequence (f_n) in $L^p[0,1]$ has a subsequence that can be decomposed into two extreme sequences (g_k) and (h_k), where the h_k's are pairwise disjoint, the g_k's are L_p-equi-integrable that is $\lim_{m(A)\to 0} \sup_k \|\chi_A g_k\|_p \to 0$ and $h_k \perp g_k$ for every $k \geq 1$. This result is generally known as the Kadec-Pełczyński subsequence decomposition and has been investigated by several authors for the cases of Banach lattices and symmetric spaces (see for instance [18] and [31]).

Motivated by the characterization of relatively weakly compact subsets of preduals of von Neumann algebras by Akemann [1], the above decomposition was studied in [9] for non-commutative L^1-spaces associated with semi-finite von Neumann algebras equipped with distinguished, faithful, normal, semi-finite traces. A more general situation on $E(\mathcal{M},\tau)$, where E is a symmetric space of functions on $(0,\infty)$ and $\mathcal{M}$ is a semi-finite von Neumann algebra, was studied in [26]. In particular, the result in [9] was generalized for $L^p(\mathcal{M},\tau)$ for all $0 < p < \infty$.

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The aim of the present paper is to provide extensions of the Kadec-Pełczyński decomposition theorem for general von Neumann algebras which are not necessarily semi-finite. There are many different methods of constructions of non-commutative L^p-spaces associated with arbitrary von Neumann algebras; for instance, those of Araki-Masuda [2], Haagerup [13], Hilsum [14], Izumi [17], Kosaki [22], Terp [29] and many others. But it is known that, for a given von Neumann algebra $\mathcal{M}$ and a fixed index p, all these L^p-spaces are isometrically isomorphic. We will consider Haagerup's L^p-spaces since they can be viewed as spaces of operators that can be embedded as subspaces of symmetric spaces of measurable operators obtained from semi-finite von Neumann algebras via crossed product (see a brief description below). Our main result is Theorem 4.1 which roughly says that any bounded sequence in $L^p(\mathcal{M})$ has a subsequence that can be splitted into two sequences; one is uniformly integrable and the other consists of elements supported by decreasing projections that converges to zero. Our initial motivation is the case $p = 1$ where $L^1(\mathcal{M})$ can simply be viewed as the predual of $\mathcal{M}$. This case allows us to get informations on copies of ℓ^1 in duals of C^*-algebras. It has been known that every non-reflexive subspace of duals of C^*-algebras contains complemented copies of ℓ^1 [25]. On the other hand, Dowling and Lennard showed in [10] that for $L^1[0,1]$, these complemented copies can be chosen to be asymptotically isometric.

Using the main decomposition for the case $p = 1$, we can conclude that every non reflexive subspace of duals of C^*-algebras contains sequences that generate complemented copies of ℓ^1 and are asymptotically isometric. As in [9] and [10], these asymptotically isometric copies of ℓ^1 yield self maps on convex bounded sets that fail to have any fixed points. These lead to a more general structural consequence that non-reflexive subspaces of duals of JB^*-triples fail the fixed point property for self-maps on closed bounded convex sets.

The paper is organized as follows: in Section 2 below, we set some preliminary background on Haagerup L^p-spaces. In particular, we provide a brief discussion on its connection to the semi-finite case and define the notion of uniformly integrable sets in these L^p-spaces. Section 3 is devoted to the proof a key result which is essentially the crusial part of the paper. We present our main results in Section 4 and finally, Section 5 is where we provide all the applications on copies of ℓ^1 on duals of C^* algebras and JB^*-triples.

Our notation and terminology are standard as may be found in [5] for Banach spaces, [21] and [27] for operator algebras.

2. Non-commutative L^p-spaces

In this section, we will describe different spaces involved and discuss some properties that will be crusial for the presentation. We will begin from the semi-finite case. We denote by $\mathcal{N}$ a semi-finite von Neumann algebra on a Hilbert space $\mathcal{H}$, with a distinguished normal,

faithful semi-finite trace τ. The identity in $\mathcal{N}$ will be denoted by $\mathbf{1}$. A closed and densely defined operator a on $\mathcal{H}$ is said to be affiliated with $\mathcal{N}$ if $ua = au$ for all unitary operator u in the commutant $\mathcal{N}'$ of $\mathcal{N}$.

A closed and densely defined operator x, affiliated with $\mathcal{N}$, is called τ-measurable if for every $\varepsilon > 0$, there exists an orthogonal projection $p \in \mathcal{N}$ such that $p(\mathcal{H}) \subseteq \mathrm{dom}(x)$, $\tau(\mathbf{1} - p) < \varepsilon$ and $xp \in \mathcal{M}$. The set of all τ-measurable operators will be denoted by $\widetilde{\mathcal{N}}$. The set $\widetilde{\mathcal{N}}$ is a $*$-algebra with respect to the strong sum, the strong product and the adjoint operation. Given a self-adjoint operator x in $\widetilde{\mathcal{N}}$ and B a Borel subset of $\mathbb{R}$, we denote by $\chi_B(x)$ the projection $\int_B 1 \; de^x$ where $e^x(\cdot)$ is the spectral measure of x. For fixed $x \in \widetilde{\mathcal{N}}$ and $t \geq 0$, we recall

$$\mu_t(x) = \inf \left\{ s \geq 0 : \tau(e^{|x|}(s, \infty)) \leq t \right\}.$$

The function $\mu_{(.)}(x) : [0, \infty) \to [0, \infty]$ is called the generalized singular value function (or decreasing rearrangement) of x. For a complete study of $\mu_{(.)}$, we refer to [12].

If E is a symmetric (r.i. for short) quasi-Banach function space on $\mathbb{R}^+$, the symmetric space of measurable operators $E(\mathcal{N}, \tau)$ is defined by setting

$$E(\mathcal{N}, \tau) := \Big\{ x \in \widetilde{\mathcal{M}} : \mu(x) \in E \Big\}$$

and

$$\|x\|_{E(\mathcal{N},\tau)} = \|\mu(x)\|_E \quad \text{for all } x \in E(\mathcal{N}, \tau).$$

The space $E(\mathcal{N}, \tau)$ is a (quasi) Banach space and is often referred to as the non-commutative version of the (quasi) Banach function space E. We remark that if $0 < p < \infty$ and $E = L^p(\mathbb{R}^+, m)$ then $E(\mathcal{N}, \tau)$ coincides with the usual non-commutative L^p-space associated to the semi-finite von Neumann algebra $\mathcal{N}$. We refer to [7], [8] and [32] for extensive background on the space $E(\mathcal{N}, \tau)$.

We now provide a short description of the Haagerup L^p-spaces. Let assume that $\mathcal{M}$ is a general von Neumann algebra (not necessarily semi-finite). Let $\mathcal{N}$ be the crossed product of $\mathcal{M}$ by the modular automorphism group $(\sigma_t)_{t\in\mathbb{R}}$ of a fixed semi-finite weight on $\mathcal{M}$. The von Neumann algebra $\mathcal{N}$ admits the dual action $(\theta_s)_{s\in\mathbb{R}}$ and a normal faithful semi-finite trace τ satisfying, $\tau \circ \theta_s = e^{-s}\tau$, $s \in \mathbb{R}$. For $1 \leq p < \infty$, the Haagerup L^p-space associated with $\mathcal{M}$ is defined by

$$L^p(\mathcal{M}) := \{x \in \widetilde{\mathcal{N}} : \theta_s(x) = e^{-s/p}x, s \in \mathbb{R}\}.$$

It is well known that there is a linear order isomorphism $\varphi \to h_\varphi$ from $\mathcal{M}_*$ onto $L^1(\mathcal{M})$. One can define a positive linear functional Tr on $L^1(\mathcal{M})$ by setting

$$Tr(h_\varphi) = \varphi(\mathbf{1}), \qquad \varphi \in \mathcal{M}_*.$$

For $1 \leq p < \infty$, the spaces $L^p(\mathcal{M})$ are Banach spaces with the norm defined by

$$\|x\|_p = (Tr(|x|^p))^{\frac{1}{p}}, \quad \text{for } x \in L^p(\mathcal{M}).$$

For complete details on the construction of $L^p(\mathcal{M})$, we refer to [28]. Also, it was shown in [12, Lemma 4.8] that if $x \in L^p(\mathcal{M})$, $1 \leq p < \infty$, then

$$\mu_t(x) = t^{-1/p}||x||_p, \quad t > 0.$$

where the singular value is relative to the canonical trace on $\mathcal{N}$.

We recall that if $1 \leq p < \infty$, then the Lorentz space $L^{p,\infty}(\mathbb{R}^+, m)$ is the set of (class of) all Lebesgue measurable functions on $\mathbb{R}^+$ with the norm

$$||f||_{p,\infty} = \sup_{t>0}\{t^{1/p}\mu_t(f)\}.$$

It is well known that if $1 < p < \infty$, then the space $L^{p,\infty}(\mathbb{R}^+, m)$ equipped with the equivalent Calderon norm given by

$$||f||_{(p,\infty)} = \sup_{t>0}\left\{t^{1/p-1}\int_0^t \mu_s(f)\, ds\right\}, \quad f \in L^{p,\infty}(\mathbb{R}^+, m),$$

is a symmetric Banach function space on $\mathbb{R}^+$ with the Fatou property. The following proposition is an immediate consequence of the above remarks.

Proposition 2.1. *If $1 < p < \infty$, then the space $L^p(\mathcal{M})$ is a closed subspace of the symmetric space of measurable operators $L^{p,\infty}(\mathcal{N}, \tau)$. Moreover if $1/q + 1/p = 1$, then*

$$||x||_p = q||x||_{(p,\infty)}$$

for all $x \in L^p(\mathcal{M})$.

Let us now extend the notion of uniform integrability to the Haagerup L^p-spaces. Following [9], we define uniform integrability in $L^p(\mathcal{M})$ as in Akemann's characterization of relatively weakly compact subsets of $\mathcal{M}_*$.

Definition 2.2. *Let $1 \leq p < \infty$ and K be a bounded subset of $L^p(\mathcal{M})$. We say that K is uniformly integrable if $\lim_{n\to\infty} \sup_{\varphi \in K} \|e_n \varphi e_n\| = 0$ for every decreasing sequence $(e_n)_n$ of projections in $\mathcal{M}$ with $e_n \downarrow_n 0$.*

We note that for $p = 1$, a subset K is uniformly integrable $L^1(\mathcal{M})$ if and only if it is relatively weakly compact.

Throughout, $\mathcal{D}$ denotes the set of all sequences of decreasing projections in $\mathcal{M}$ that converges to zero; $\mathcal{D} := \{(e_n)_n;$ the e_n's are projections in $\mathcal{M}$ and $e_n \downarrow_n 0\}$. Also for any subset K of $L^p(\mathcal{M})$, $|K|$ denotes the set of all modudi of elements of K; $|K| := \{|x|; x \in K\}$.

Fact 1. *If $x \in L^p(\mathcal{M})$ and $y \in L^p(\mathcal{M})$ are such that $x \perp y$ (i.e. $(\operatorname{supp} x) \perp (\operatorname{supp} y)$) then $||x+y||^p = ||x||^p + ||y||^p$.*

Proof. $||x||^p = Tr(|x|^p)$ and if $x \perp y$ as elements of $\mathcal{N}$, $|x+y|^p = |x|^p + |y|^p$ and therefore $||x+y||^p = Tr(|x|^p + |y|^p) = ||x||^p + ||y||^p$. □

Fact 2. *If $x \in L^p(\mathcal{M})$ and e is a projection in $\mathcal{M}$, then $\|x\|^p \geq \|exe\|^p + \|(1-e)x(1-e)\|^p$.*

Proof. Set $u = 2e - 1$. It is clear that $u \in \mathcal{M}$ is unitary and $exe + (1-e)x(1-e) = \frac{1}{2}(x + uxu^*)$. It follows that $\|exe + (1-e)x(1-e)\|^p \leq \|x\|^p$ and hence $\|exe\|^p + \|(1-e)x(1-e)\|^p \leq \|x\|^p$. □

We finish this section with the following two lemmas which can be proved using similar arguments as in the semi-finite case ([9], [26]) and will be used in the sequel. Details are left to the readers.

Lemma 2.3. *Let $1 \leq p < \infty$, $(p_n)_n \in \mathcal{D}$ and K be a bounded subset of $L^p(\mathcal{M})$ such that for each $n_0 \geq 1$, the sets $(\mathbf{1} - p_{n_0})K$ and $|K(\mathbf{1} - p_{n_0})|$ are uniformly integrable. Then K is uniformly integrable if and only if $\lim\limits_{n\to\infty} \sup_{\varphi \in K} \|p_n \varphi p_n\| = 0$.*

Lemma 2.4. *Let $1 \leq p < \infty$, $(\varphi_n)_n$ be a bounded sequence in $L^p(\mathcal{M})$ and $(p_n)_n \in \mathcal{D}$. Assume that $\lim\limits_{n\to\infty} \sup\limits_k \|p_n \varphi_k p_n\| = \gamma > 0$ then there exists a subsequence (φ_{k_n}) so that $\lim\limits_{n\to\infty} \|p_n \varphi_{k_n} p_n\| = \gamma$.*

3. Preliminary Results

This section is devoted to the proof of Theorem 3.1 below which is the key result that we will use to prove our main theorem. We remark that the case of finite von-Neumann algebras can be obtained with minor changes from the proof of the commutaive case (see [9]).

Theorem 3.1. *Let $\mathcal{M}$ be a σ-finite von-Neumann algebra and $1 \leq p < \infty$. Assume that K is a subset of the positive part of the unit ball of $L^p(\mathcal{M})$ that is not uniformly integrable. Then there exists a sequence $(\varphi_n)_n \subset K$ and $(f_n)_n \in \mathcal{D}$ such that:*

$$\sup\left\{\lim_{n\to\infty} \sup_{k\in\mathbb{N}} \|e_n \varphi_k e_n\|;\ (e_n)_n \in \mathcal{D}\right\} = \lim_{n\to\infty} \sup_{k\in\mathbb{N}} \|f_n \varphi_k f_n\| > 0.$$

Lemma 3.2. *Let $\mathcal{N}$ be a semi-finite von Neumann algebra with distinguished faithful normal semi-finite trace τ as above and E be a symmetric quasi-Banach function space on $(0,\infty)$. If $x \in E(\mathcal{N}, \tau)$ and $u \in \mathcal{N}$ then*

$$||xu||_E \leq ||x||_E^{\frac{1}{2}} \cdot ||u^*|x|u||_E^{\frac{1}{2}} \leq ||x||_E^{\frac{3}{4}} \cdot ||uu^*|x|uu^*||_E^{\frac{1}{4}}.$$

Proof. Let $x = v|x|$ be the polar decomposition of x. Then

$$\begin{aligned} \|xu\| = ||v|x|u|| &\leq |||x|u|| = |||x|^{\frac{1}{2}}|x|^{\frac{1}{2}}u|| \\ &\leq |||x|^{\frac{1}{2}}||_{E^{(2)}} \cdot |||x|^{\frac{1}{2}}u||_{E^{(2)}} \\ &= \|x\|_E^{\frac{1}{2}} \cdot \|u^*|x|u\|_E^{\frac{1}{2}}. \end{aligned}$$

For the second inequality,

$$\begin{aligned} \|xu\| &\leq ||x||_E^{\frac{1}{2}} \cdot |||x|^{\frac{1}{2}}u||_{E^{(2)}} \\ &= ||x||_E^{\frac{1}{2}} \cdot ||u^*|x|^{\frac{1}{2}}||_{E^{(2)}} \\ &\leq ||x||_E^{\frac{1}{2}} \cdot |||x|^{\frac{1}{2}}uu^*|x|^{\frac{1}{2}}||_E^{\frac{1}{2}} \\ &\leq ||x||_E^{\frac{1}{2}} \left(\left\||x|^{\frac{1}{2}}uu^*\right\|_{E^{(2)}} \cdot \left\||x|^{\frac{1}{2}}\right\|_{E^{(2)}} \right)^{\frac{1}{4}} \\ &= ||x||_E^{\frac{1}{2}} \cdot ||uu^*|x|uu^*||_E^{\frac{1}{4}} \cdot ||x||_E^{\frac{1}{4}} \\ &= ||x||_E^{\frac{3}{4}} \cdot ||uu^*|x|uu^*||_E^{\frac{1}{4}}. \end{aligned}$$

□

Lemma 3.2 shows in particular that if $1 \leq p < \infty$, $x \in L^p(\mathcal{M})$ and $u \in \mathcal{M}$ then

$$||xu|| \leq ||x||^{\frac{3}{4}} \cdot ||uu^*|x|uu^*||^{\frac{1}{4}}.$$

Lemma 3.3. *Let $\gamma > 0$ and $(\varphi_k)_k$ be a sequence in the positive part of the unit ball of $L^p(\mathcal{M})$. If there exists a sequence $(a_n)_n$ in the unit ball of $\mathcal{M}$ with $a_n \downarrow_n 0$ and such that $\lim_{n\to\infty} \sup_k ||a_n\varphi_k a_n|| \geq \gamma$. Then for every $\varepsilon > 0$, there exists a sequence $(s_n)_n$ of projections with:*

(i) $s_n \leq s_1$ *for every* $n \geq 1$*;*
(ii) $s_n \to 0$ *for the strong operator topology;*
(iii) *for every* $n_0 \in \mathbb{N}$, $\lim_{n\to\infty} \sup_k \|(s_{n_0}a_n s_{n_0})\varphi_k(s_{n_0}a_n s_{n_0})\| \geq \gamma - \varepsilon$.

Proof. Fix $\delta > 0$ with $\delta \leq (\varepsilon/8)^2$ and define the sequence of projections as follows:

$$\begin{cases} s_1 := \chi_{(\delta,1)}(a_1) & \textit{and} \\ s_n := \chi_{(\delta,1)}(s_1 a_n s_1) & \textit{for } n \geq 2. \end{cases}$$

Clearly s_n is a subprojection of the support of $s_1 a_n s_1$ so $s_n \leq s_1$. Also $\delta s_n \leq s_n(s_1 a_n s_1)s_n$, and since s_n and $s_1 a_n s_1$ are commuting operators, $\delta s_n \leq s_n(s_1 a_n s_1)s_n \leq s_1 a_n s_1$ and therefore $s_n \to 0$ so (i) and (ii) are verified.

Claim: Let $n_0 \in \mathbb{N}$ and $n \geq n_0$, for every $\varphi \in L^p(\mathcal{M})$ with $\|\varphi\| \leq 1$, $\|\varphi a_n(1-s_{n_0})\| \leq 2\delta^{1/2}$. Similarily, $\|\varphi(1-s_{n_0})a_n\| \leq 2\delta^{1/2}$.

To see this claim, it is enough to notice that

$$\begin{aligned}
\|\varphi a_n(1-s_{n_0})\| &\leq \|\varphi a_n s_1(1-s_{n_0})\| + \|\varphi a_n(1-s_1)\| \\
&\leq \|\varphi\| \cdot \|a_n s_1(1-s_{n_0})\|_\infty + \|\varphi\| \cdot \|a_n(1-s_1)\|_\infty \\
&\leq \|(1-s_{n_0})s_1 a_n^2 s_1(1-s_{n_0})\|_\infty^{1/2} + \|(1-s_1)a_n^2(1-s_1)\|_\infty^{1/2} \\
&\leq \|(1-s_{n_0})s_1 a_n s_1(1-s_{n_0})\|_\infty^{1/2} + \|(1-s_1)a_n(1-s_1)\|_\infty^{1/2}
\end{aligned}$$

and since $(a_n)_n$ is a decreasing sequence and $1 \leq n_0 \leq n$, we get

$$\|\varphi a_n(1-s_{n_0})\| \leq \|(1-s_{n_0})s_1 a_{n_0} s_1(1-s_{n_0})\|_\infty^{1/2} + \|(1-s_1)a_1(1-s_1)\|_\infty^{1/2} \leq 2\delta^{1/2}.$$

A similar estimate can be established for $\|\varphi(1-s_{n_0})a_n\|$ which verifies the claim.

To complete the proof, let $\varphi \in L^p(\mathcal{M})$, $\|\varphi\| \leq 1$. For $n \geq n_0$, we can write $a_n\varphi a_n$ as:

$$a_n\varphi a_n = (s_{n_0}a_n s_{n_0})\varphi a_n + s_{n_0}a_n(1-s_{n_0})\varphi a_n + (1-s_{n_0})a_n\varphi a_n$$

and using the claim above, $\|a_n\varphi a_n\| \leq \|(s_{n_0}a_n s_{n_0})\varphi a_n\| + 4\delta^{1/2}$. A similar estimate would give $\|(s_{n_0}a_n s_{n_0})\varphi a_n\| \leq \|(s_{n_0}a_n s_{n_0})\varphi(s_{n_0}a_n s_{n_0})\| + 4\delta^{1/2}$ and combining these two estimates, we get

$$\|a_n\varphi a_n\| \leq \|(s_{n_0}a_n s_{n_0})\varphi(s_{n_0}a_n s_{n_0})\| + 8\delta^{1/2}.$$

This shows that

$$\lim_{n\to\infty} \sup_k \|(s_{n_0}a_n s_{n_0})\varphi_k(s_{n_0}a_n s_{n_0})\| \geq \gamma - 8\delta^{1/2} \geq \gamma - \varepsilon.$$

The proof is complete. □

The next result shows that using projections in the definition of uniform integrability is not essential. One can use elements of the positive part of the unit ball of $\mathcal{M}$.

Proposition 3.4. *Let $\gamma > 0$ and $(\varphi_k)_k$ be a sequence in the positive part of the unit ball of $L^p(\mathcal{M})$. If there exists a sequence $(a_n)_n$ in the unit ball of $\mathcal{M}$ with $a_n \downarrow_n 0$ and such that* $\lim_{n\to\infty}\sup_k ||a_n\varphi_k a_n|| \geq \gamma$. *Then for every $\varepsilon > 0$, there exists a sequence $(p_n)_n \in \mathcal{D}$ with $p_n \leq supp(a_1)$ for all $n \geq 1$ and such that* $\lim_{n\to\infty}\sup_k ||p_n\varphi_k p_n|| \geq \gamma - \varepsilon$.

Proof. The sequence (p_n) will be constructed inductively. Let $(\varepsilon_j)_j$ be a sequence in the open interval $(0, \varepsilon)$ such that $\sum_{j=1}^\infty \varepsilon_j = \varepsilon$ and ω_0 be a faithful state in $\mathcal{M}_*$.

By Lemma 3.3, one can choose a sequence of projections $(s_n^{(1)})_n$ with $s_n^{(1)} \leq s_1^{(1)}$ for every $n \geq 1$, $s_n^{(1)} \to 0$ (as n tends to ∞) satisfying the conclusion of Lemma 3.3 for $(a_n)_n$, γ and ε_1.

Choose $n_1 \geq 1$ such that $\omega_0 \left(s^{(1)}_{n_1}\right) \leq 1/2$. From (iii) of Lemma 3.3,

$$\lim_{n\to\infty} \sup_k \|(s^{(1)}_{n_1} a_n s^1_{n_1})\varphi_k(s^{(1)}_{n_1} a_n s^{(1)}_{n_1})\| \geq \gamma - \varepsilon_1.$$

Reapplying Lemma 3.3, on $(a^{(2)}_n)_n = (s^{(1)}_{n_1} a_n s^{(1)}_{n_1})_n$, $\gamma - \varepsilon_1$ and ε_2, one would get a sequence of projections $(s^{(2)}_n)_n$ with $s^{(2)}_n \leq s^{(1)}_{n_1}$ for every $n \geq 1$, $s^{(2)}_n \to 0$ (as n tends to infinity). As above, on can choose n_2 such that $\omega_0 \left(s^{(2)}_{n_2}\right) \leq 1/2^2$ and

$$\lim_{n\to\infty} \sup_k \|(s^{(2)}_{n_2} a_n s^{(2)}_{n_2})\varphi_k(s^{(2)}_{n_2} a_n s^{(2)}_{n_2})\| \geq \gamma - \varepsilon_1 - \varepsilon_2.$$

The induction is clear, repeating the argument above would give a decreasing sequence of projections $s^{(1)}_{n_1} \geq s^{(2)}_{n_2} \geq \cdots \geq s^{(j)}_{n_j} \geq \cdots$ so that for every $j \geq 1$, $\omega_0 \left(s^{(j)}_{n_j}\right) \leq 1/2^j$ and

$$\lim_{n\to\infty} \sup_k \|(s^{(j)}_{n_j} a_n s^{(j)}_{n_j})\varphi_k(s^{(j)}_{n_j} a_n s^{(j)}_{n_j})\| \geq \gamma - \sum_{i=1}^{j} \varepsilon_i.$$

If for every $j \geq 1$, we set $p_j = s^{(j)}_{n_j}$ then $(p_j)_j$ belongs to $\mathcal{D}$ and

$$\sup_k \|(p_j a_j p_j)\varphi_k(p_j a_j p_j)\| \geq \gamma - \sum_{i=1}^{j} \varepsilon_i$$

which shows that

$$\lim_{j\to\infty} \sup_k \|(p_j a_j p_j)\varphi_k(p_j a_j p_j)\| \geq \gamma - \varepsilon$$

and since $\|p_j a_j\|_\infty \leq 1$, the desired conclusion follows. $\square$

Proposition 3.5. *Let K be as in the statement of Theorem 3.1. There exists a sequence $(\varphi_k)_k$ in K such that*

$$\sup\left\{\lim_{n\to\infty} \sup_{k\in\mathbb{N}} \|e_n\varphi_k e_n\|;\ (e_n)_n \in \mathcal{D}\right\} = \sup\left\{\underline{\lim}_{\,n\to\infty} \|e_n\varphi_n e_n\|;\ (e_n)_n \in \mathcal{D}\right\} > 0.$$

Proof. Set $\alpha_0 := \sup\left\{\lim_{n\to\infty}\sup_{\varphi\in K} \|e_n\varphi e_n\|;\ (e_n) \in \mathcal{D}\right\}$ and let $(\varepsilon_j)_j$ be a subset of the open interval $(0,1)$ such that $\Pi_{j=1}^{\infty}(1-\varepsilon_j) > 0$.

Since $\alpha_0 > 0$, one can choose a sequence $(y_n)_n$ in K and $(e^{(1)}_n)_n \in \mathcal{D}$ such that

$$\lim_{n\to\infty} \sup_{k\in\mathbb{N}} \|e^{(1)}_n y_k e^{(1)}_n\| \geq \alpha_0(1-\varepsilon_1).$$

A further subsequence $(y^{(1)}_k)_k \subset (y_k)$ can be chosen so that

$$\lim_{n\to\infty} \|e^{(1)}_n y^{(1)}_n e^{(1)}_n\| \geq \alpha_0(1-\varepsilon_1).$$

Set $\alpha_1 := \sup\left\{\lim_{n\to\infty}\sup_{k\in\mathbb{N}}\|e_n y_k^{(1)} e_n\|; (e_n)\in\mathcal{D}\right\}$. It is clear that $\alpha_1 \geq \alpha_0(1-\varepsilon_1)$ and as above a sequence $(y_k^{(2)})_{k\geq 1} \subseteq (y_k^{(1)})_k$ can be chosen so that

$$\lim_{n\to\infty}\|e_n^{(2)} y_n^{(2)} e_n^{(2)}\| \geq \alpha_1(1-\varepsilon_2).$$

Inductively, one can construct sequences $(y_n)_n \supseteq (y_n^{(1)})_n \supseteq (y_n^{(2)})_n \supseteq \dots (y_n^{(j)})_n \supseteq \dots$ in K and sequences $(e_n^{(1)})_n, (e_n^{(2)})_n, \dots, (e_n^{(j)})_n, \dots$ in $\mathcal{D}$ so that for every $j \geq 1$,

$$\lim_{n\to\infty}\|e_n^{(j)} y_n^{(j)} e_n^{(j)}\| \geq \alpha_{j-1}(1-\varepsilon_j).$$

Let $(\varphi_n)_n$ be the diagonal sequence obtained from $(y_n^{(j)})_n$, $j \geq 1$. For every $j \geq 1$, $(\varphi_n)_{n\geq j}$ is a subsequence of $(y_n^{(j)})_{n\geq 1}$ so

$$\underline{\lim}_{\,n\to\infty}\|e_n^{(j)}\varphi_n e_n^{(j)}\| \geq \alpha_{j-1}(1-\varepsilon_j)$$

and

$$\sup\left\{\lim_{n\to\infty}\sup_{k\in\mathbb{N}}\|e_n\varphi_k e_n\|; (e_n)\in\mathcal{D}\right\} \leq \alpha_j.$$

We note that $\alpha_{j-1} \geq \alpha_j \geq \alpha_{j-1}(1-\varepsilon_j)$ so for every $j \geq 1$,

$$\begin{aligned}\sup\left\{\lim_{n\to\infty}\sup_{k\in\mathbb{N}}\|e_n\varphi_k e_n\|; (e_n)\in\mathcal{D}\right\} &\leq \alpha_j \\ &\leq \alpha_j(1-\varepsilon_{j+1})\frac{1}{1-\varepsilon_{j+1}} \\ &\leq \frac{1}{1-\varepsilon_{j+1}}\,\underline{\lim}_{\,n\to\infty}\|e_n^{(j+1)}\varphi_n e_n^{(j+1)}\|\end{aligned}$$

which implies that

$$\sup\{\lim_{n\to\infty}\sup_{k\in\mathbb{N}}\|e_n\varphi_k e_n\|;\ (e_n)_n\in\mathcal{D}\} \leq \frac{1}{1-\varepsilon_{j+1}}\sup\left\{\underline{\lim}_{\,n\to\infty}\|e_n\varphi_n e_n\|;\ (e_n)_n\in\mathcal{D}\right\}.$$

Taking the limit as j goes to ∞,

$$\sup\left\{\lim_{n\to\infty}\sup_{k\in\mathbb{N}}\|e_n\varphi_k e_n\|;\ (e_n)_n\in\mathcal{D}\right\} \leq \sup\left\{\underline{\lim}_{\,n\to\infty}\|e_n\varphi_n e_n\|;\ (e_n)_n\in\mathcal{D}\right\}.$$

The other inequality is trivial.

To check that $\sup\left\{\underline{\lim}_{\,n\to\infty}\|e_n\varphi_n e_n\|;\ (e_n)_n\in\mathcal{D}\right\} > 0$, it is plain that

$$\underline{\lim}_{\,n\to\infty}\|e_n^{(j)}\varphi_n e_n^{(j)}\| \geq \alpha_{j-1}(1-\varepsilon_j) \geq \alpha_1\Pi_{j=2}^{\infty}(1-\varepsilon_j) > 0.$$

The proof of the proposition is complete. $\square$

Proof of Theorem 3.1

Let (φ_n) be the sequence in K obtained from Proposition 3.5 ; i.e. $(\varphi_n)_n$ is the sequence in K satisfying:

$$\sup\{\underline{\lim}_{n\to\infty}\|e_n\varphi_n e_n\|;\ (e_n)_n\in\mathcal{D}\} = \sup\left\{\lim_{n\to\infty}\sup_{k\in\mathbb{N}}\|e_n\varphi_k e_n\|;\ (e_n)_n\in\mathcal{D}\right\} := \alpha > 0$$

Claim: α is attained.

Assume the opposite i.e. for every $(p_n)_n\in\mathcal{D}$, $\underline{\lim}_{n\to\infty}\|p_n\varphi_n p_n\| < \alpha$.

Inductively, we will construct sequences of integers and projections in $\mathcal{M}$ satisfying the following conditions:

$$m_1 \leq m_2 \leq \cdots \leq m_j \leq \cdots \text{ a sequence in } \mathbb{N}; \tag{3.1}$$

$$n_1 < n_2 < \cdots < n_j < \cdots \text{ infinite sequence in } \mathbb{N}; \tag{3.2}$$

sequences $(p_n^{(1)})_{n\geq 1}, (p_n^{(2)})_{n\geq 1}, \cdots$ in $\mathcal{D}$ such that for every $j \geq 2$ and every $n \geq 2$,

$$p_n^{(j)} \perp \sum_{k=1}^{j-1} p_{n_k}^{(k)}; \tag{3.3}$$

if we set $(f_n^{(1)})_{n\geq 1} = (p_n^{(1)})_{n\geq 1}$ and

$$f_n^{(j)} = \begin{cases} f_n^{(j-1)} \vee p_{n_{j-1}}^{(j)} & n < n_{j-1} \\ f_n^{(j-1)} + p_n^{(j)} & n \geq n_{j-1} \end{cases} \tag{3.4}$$

then

$$\lim_{n\to\infty}\sup_{k\in\mathbb{N}}\|f_n^{(j)}\varphi_k f_n^{(j)}\| \geq \alpha\left(1-\frac{1}{2^{m_j-1}}\right), \tag{3.5}$$

$$\sup_{k\in\mathbb{N}}\|f_{n_j}^{(j)}\varphi_k f_{n_j}^{(j)}\| < \alpha\left(1-\frac{1}{2^{m_j}}\right) \tag{3.6}$$

and

$$\underline{\lim}_{n\to\infty}\|f_n^{(j)}\varphi_n f_n^{(j)}\|^p \geq \underline{\lim}_{n\to\infty}\|f_n^{(j-1)}\varphi_n f_n^{(j-1)}\|^p + \frac{\alpha^{4p}}{(2^{m_{j-1}+3})^{4p}}. \tag{3.7}$$

Fix a sequence $(p_n^{(1)})_{n\geq 1}\in\mathcal{D}$ such that $\underline{\lim}_{n\to\infty}\|p_n^{(1)}\varphi_n p_n^{(1)}\| \geq \alpha(1-\frac{1}{2^2})$ and choose $m_1\in\mathbb{N}$ such that

$$\alpha\left(1-\frac{1}{2^{m_1-1}}\right) \leq \lim_{n\to\infty}\sup_{k\in\mathbb{N}}\|p_n^{(1)}\varphi_k p_n^{(1)}\| < \alpha\left(1-\frac{1}{2^{m_1}}\right)$$

(such m_1 exists since α is not attained).

Assume that the construction is done for $1, 2, \cdots, (j-1)$. By the definition of α, one can choose $(q_n)_n \in \mathcal{D}$ so that $\underline{\lim}_{n\to\infty}\|q_n\varphi_n q_n\| > \alpha(1 - \frac{1}{2^{m_{j-1}+1}})$. Writing $q_n\varphi_n q_n$ in the form

$$q_n\varphi_n q_n = q_n f^{(j-1)}_{n_{j-1}}\varphi_n f^{(j-1)}_{n_{j-1}} q_n + q_n f^{(j-1)}_{n_{j-1}}\varphi_n(1 - f^{(j-1)}_{n_{j-1}})q_n + q_n(1 - f^{(j-1)}_{n_{j-1}})\varphi_n q_n,$$

one can see that

$$\|q_n\varphi_n q_n\| \le \|f^{(j-1)}_{n_{j-1}}\varphi_n f^{(j-1)}_{n_{j-1}}\| + 2\|\varphi_n(1 - f^{(j-1)}_{n_{j-1}})q_n\|.$$

Applying Lemma 3.2 for $x = \varphi_n$ and $u = (1 - f^{(j-1)}_{n_{j-1}})q_n$, we get

$$\begin{aligned}\|q_n\varphi_n q_n\| \le \|&f^{(j-1)}_{n_{j-1}}\varphi_n f^{(j-1)}_{n_{j-1}}\| + \\ &2\|\varphi_n\|^{\frac{3}{4}} \cdot \|(1 - f^{(j-1)}_{n_{j-1}})q_n(1 - f^{(j-1)}_{n_{j-1}})\varphi_n(1 - f^{(j-1)}_{n_{j-1}})q_n(1 - f^{(j-1)}_{n_{j-1}})\|^{\frac{1}{4}}.\end{aligned}$$

Applying (3.6) for $(j-1)$ gives

$$\|q_n\varphi_n q_n\| \le \alpha(1 - \frac{1}{2^{m_{j-1}}}) + 2\|(1 - f^{(j-1)}_{n_{j-1}})q_n(1 - f^{(j-1)}_{n_{j-1}})\varphi_n(1 - f^{(j-1)}_{n_{j-1}})q_n(1 - f^{(j-1)}_{n_{j-1}})\|^{\frac{1}{4}}.$$

Taking the limit (as n tends to ∞),

$$\alpha(1-\frac{1}{2^{m_{j-1}+1}}) \le \alpha(1-\frac{1}{2^{m_{j-1}}})+2\,\underline{\lim}_{n\to\infty}\|(1-f^{(j-1)}_{n_{j-1}})q_n(1-f^{(j-1)}_{n_{j-1}})\varphi_n(1-f^{(j-1)}_{n_{j-1}})q_n(1-f^{(j-1)}_{n_{j-1}})\|^{\frac{1}{4}}.$$

which implies that

$$\underline{\lim}_{n\to\infty}\|(1 - f^{(j-1)}_{n_{j-1}})q_n(1 - f^{(j-1)}_{n_{j-1}})\varphi_n(1 - f^{(j-1)}_{n_{j-1}})q_n(1 - f^{(j-1)}_{n_{j-1}})\| \ge \frac{\alpha^4}{(2^{m_{j-1}+2})^4}.$$

If we set $a^{(j)}_n = (1 - f^{(j-1)}_{n_{j-1}})q_n(1 - f^{(j-1)}_{n_{j-1}})$ then $a^{(j)}_n \downarrow_n 0$ and

$$\underline{\lim}_{n\to\infty}\|a^{(j)}_n\varphi_n a^{(j)}_n\| \ge \frac{\alpha^4}{(2^{m_{j-1}+2})^4}.$$

Applying Proposition 3.4 for $(\varphi_n)_n$, $(a^{(j)}_n)_n$, $\gamma = \frac{\alpha^4}{(2^{m_{j-1}+2})^4}$ and $\varepsilon = \frac{\alpha^4}{(2^{m_{j-1}+2})^4} - \frac{\alpha^4}{(2^{m_{j-1}+3})^4}$ would provide a sequence $(p^{(j)}_n) \in \mathcal{D}$ such that

$$\underline{\lim}_{n\to\infty}\|p^{(j)}_n\varphi_n p^{(j)}_n\| \ge \frac{\alpha^4}{(2^{m_{j-1}+3})^4}.$$

Since $p^{(j)}_n \le \mathrm{supp}(a^{(j)}_1) \le 1 - f^{(j-1)}_{n_{j-1}}$, it is clear that $p^{(j)}_n \perp f^{(j-1)}_{n_{j-1}}$ for every $n \ge 1$ so (3.3) is verified.

If we define $(f^{(j)}_n)$ as in (3.4) then appropriate $m_j \ge m_{j-1}$ and $n_j > n_{j-1}$ can be choosen so that (3.5) and (3.6) would be satisfied.

Now since $p^{(j)}_n + f^{(j-1)}_n = f^{(j)}_n$ for $n \ge n_j$,

$$\begin{aligned}\|f_n^{(j)}\varphi_n f_n^{(j)}\|^p &\geq \|f_n^{(j-1)}\varphi_n f_n^{(j-1)}\|^p + \|p_n^{(j)}\varphi_n p_n^{(j)}\|^p \\ &\geq \|f_n^{(j-1)}\varphi_n f_n^{(j-1)}\|^p + \frac{\alpha^{4p}}{(2^{m_{j-1}+3})^{4p}}\end{aligned}$$

and (3.7) is verified. The construction is done.

To complete the proof of the theorem, we note from (3.7) that

$$\underline{\lim}_{n\to\infty}\|f_n^{(j)}\varphi_n f_n^{(j)}\|^p \geq \underline{\lim}_{n\to\infty}\|f_n^{(1)}\varphi_n f_n^{(1)}\|^p + \alpha^{4p}\sum_{k=1}^{j-1}\frac{1}{(2^{m_k+3})^{4p}}$$

So the series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{(2^{m_k+3})^{4p}}$ is convergent. In particular $\lim_{k\to\infty} m_k = \infty$.

We note from (3.4) that every $j \geq 1$ and $n \geq n_{j-1}$, $f_n^{(j)} = \displaystyle\sum_{k=1}^{j} p_n^{(k)} = \vee_{k=1}^{j} p_n^{(k)}$; this is the case since (n_j) is increasing so if $n \geq n_{j-1}$ then $n \geq n_{l-1}$ for all $l \leq j$ and hence

$$\begin{aligned}f_n^{(j)} &= f_n^{(j-1)} + p_n^{(j)} \\ &= f_n^{(j-2)} + p_n^{(j-1)} + p_n^{(j)} \\ &= \sum_{k=1}^{j} p_n^{(k)}\end{aligned}$$

and all the $(p_n^{(k)})_{1\leq k\leq j}$ are mutually disjoint.

Now choose an increasing sequence (k_j) so that $k_j > \max(k_{j-1}, n_{j-1})$, $\omega_0(f_{k_j}^{(j)}) < \frac{1}{2^j}$ and

$$\alpha\Big(1 - \frac{1}{2^{m_j-2}}\Big) \leq \sup_{k\in\mathbb{N}}\|f_{k_j}^{(j)}\varphi_k f_{k_j}^{(j)}\| \tag{3.8}$$

(this last condition is possible from (3.5)).

Claim: $\{f_{k_j}^{(j)}; j \geq 1\}$ *is a commuting family of projections in* $\mathcal{M}$.

In fact for each $j \geq l$, $f_{k_j}^{(j)} = \sum_{k=1}^{j} p_{k_j}^{(k)}$ and $f_{k_l}^{(l)} = \sum_{k=1}^{l} p_{k_l}^{(k)}$. For each $1 \leq k \leq l$, $p_{k_l}^{(k)} \geq p_{k_j}^{(k)}$ and $p_{k_l}^{(k)} \perp \displaystyle\sum_{s=1;s\neq k}^{j} p_{k_j}^{(s)}$, hence

$$f_{k_j}^{(j)} f_{k_l}^{(l)} = f_{k_l}^{(l)} f_{k_j}^{(j)} = \sum_{k=1}^{l} p_{k_j}^{(k)}$$

and the claim follows.

Set $\mathcal{S}$ to be a maximal abelian von Nemann subalgebra of $\mathcal{M}$ generated by $\left\{ f_{k_j}^{(j)} \,; j \geq 1 \right\}$. Since $\mathcal{S}$ is abelian, ω_0 restricted to $\mathcal{S}$ is a faithful tracial state on $\mathcal{S}$.

Set $p_n := \bigvee_{j \geq n} f_{k_j}^{(j)}$ (where the supremum is taken in $\mathcal{S}$). It is clear that

$$\omega_0(p_n) \leq \sum_{j=n}^{\infty} \omega_0(f_{k_j}^{(j)}) \leq \sum_{j=n}^{\infty} \frac{1}{2^j}$$

so $\omega_0(p_n) \to 0$ which shows that $(p_n)_n \in \mathfrak{D}$. Moreover, since $p_n \geq f_{k_n}^{(n)}$ for all $n \geq 1$, condition (3.8) implies that

$$\sup_{k \in \mathbb{N}} \|p_n \varphi_k p_n\| \geq \alpha(1 - \frac{1}{2^{m_n - 2}}).$$

This would show that $\lim_{n \to \infty} \sup_{k \in \mathbb{N}} \|p_n \varphi_k p_n\| = \alpha$.

This is a contradiction with the initial assumption that α is not attained. The proof is complete.

□

4. Main Result

The main results in this section are Theorem 4.1 and Theorem 4.4 which generalize the classical Kadec-Pełczyński subsequence decomposition to bounded sequences in the Haagerup L^p-spaces.

Theorem 4.1. *Let $\mathcal{M}$ be a von Nemann algebra, $1 \leq p < \infty$ and $(\varphi_n)_n$ be a bounded sequence in $L^p(\mathcal{M})$. Then there exist a subsequence $(\varphi_{n_k})_k$ of $(\varphi_n)_n$, two bounded sequences (y_k), (z_k) in $L^p(\mathcal{M})$ and a decreasing sequence of projections $e_k \downarrow_k 0$ in $\mathcal{M}$ such that:*

(i) $\varphi_{n_k} = y_k + z_k$ *for all* $k \geq 1$;
(ii) $\{y_k,\ k \geq 1\}$ *is uniformly integrable in* $L^p(\mathcal{M})$;
(iii) $z_k = e_k z_k e_k$ *for all* $k \geq 1$.

Proof. We will assume first that $\mathcal{M}$ is σ-finite. Without loss of generality, we can and do assume that $\|\varphi_n\| \leq 1$ for all $n \geq 1$ and $\{\varphi_n, n \geq 1\}$ is not uniformly integrable. We will show that there exist a sequence (n_k) in $\mathbb{N}$ and $(e_k)_k \in \mathfrak{D}$ such that the bounded set $\{\varphi_{n_k} - e_k\ \varphi_{n_k}\ e_k; k \geq 1\}$ is uniformly integrable in $L^p(\mathcal{M})$.

By Theorem 3.1, there exists a subsequence of $(\varphi_n)_n$ (which we will denote again by $(\varphi_n)_n$) and $(e_n)_n \in \mathfrak{D}$ such that

$$\sup\{\lim_{n\to\infty}\sup_{k\in\mathbb{N}}\|f_n(|\varphi_k|+|\varphi_k^*|)f_n\|;(f_n)\in\mathcal{D}\}$$
$$=\lim_{n\to\infty}\sup_{k\in\mathbb{N}}\|e_n(|\varphi_k|+|\varphi_k^*|)e_n\|=\alpha>0.$$

Choose a subsequence $(\varphi_{n_k})_k\subset(\varphi_k)$ so that

$$\lim_{k\to\infty}\|e_k\ \varphi_{n_k}\ e_k\|=\alpha. \tag{4.1}$$

Set $u_k:=\varphi_{n_k}$ and $v_k:=u_k-e_ku_ke_k$ for all $k\geq 1$.

Claim: The set $V=\{v_k;\ k\in\mathbb{N}\}$ is uniformly integrable in $L^p(\mathcal{M})$.

To see this claim, we will first prove the following intermediate lemma:

Lemma 4.2. *Let $n_0\in\mathbb{N}$, $(1-e_{n_0})V$ and $|V(1-e_{n_0})|$ are uniformly integrable subsets of $L^p(\mathcal{M})$.*

We will show that $|V(1-e_{n_0})|$ is uniformly integrable. Assume the opposite. There exists $(f_n)_n\in\mathcal{D}$ such that

$$\lim_{n\to\infty}\sup_{k\in\mathbb{N}}\|f_n|v_k(1-e_{n_0})|f_n\|>0.$$

From this, there would exists $(p_n)_n\in\mathcal{D}$ with $p_n\leq 1-e_{n_0}$ and such that

$$\lim_{n\to\infty}\sup_{k\in\mathbb{N}}\|p_n(|u_k|+|u_k^*|p_n)\|>0.$$

In fact, for each $k\geq 1$, if we denote by ω_k the partial isometry in $\mathcal{M}$ so that $|v_k(1-e_{n_0})|=\omega_kv_k(1-e_{n_0})$, then

$$\|f_n|v_k(1-e_{n_0})|f_n\|=\|f_n\omega_kv_k(1-e_{n_0})f_n\|\leq\|v_k(1-e_{n_0})f_n\|$$

Note that for $k\geq n_0$, $e_k(1-e_{n_0})=0$ so $\|f_n|v_k(1-e_{n_0})|f_n\|\leq\|u_k(1-e_{n_0})f_n\|$ and by Lemma 3.2,

$$\|f_n|v_k(1-e_{n_0})|f_n\|\leq\|u_k\|^{\frac{3}{4}}\cdot\|(1-e_{n_0})f_n(1-e_{n_0})|u_k|(1-e_{n_0})f_n(1-e_{n_0})\|^{\frac{1}{4}}$$

which shows that

$$\|f_n|v_k(1-e_{n_0})|f_n\|\leq\|(1-e_{n_0})f_n(1-e_{n_0})|u_k|(1-e_{n_0})f_n(1-e_{n_0})\|^{\frac{1}{4}}.$$

Let $a_n=(1-e_{n_0})f_n(1-e_{n_0})$. It is clear that $a_n\downarrow_n 0$ and using Proposition 3.4, we conclude that there exists $(p_n)_n\in\mathcal{D}$, $p_n\leq 1-e_{n_0}$ such that $\lim_{n\to\infty}\sup_{k\in\mathbb{N}}\|p_n|u_k|p_n\|>0$. In particular:

$$\lim_{n\to\infty}\sup_{k\in\mathbb{N}}\|p_n(|u_k|+|u_k^*|)p_n\|>0.$$

Now choose a subsequence $(k_j)\subseteq\mathbb{N}$ so that there exists $\gamma>0$ satisfying

$$\lim_{j\to\infty} \|p_j(|u_{k_j}| + |u_{k_j}^*|)p_j\| = \gamma > 0. \tag{4.2}$$

Since $p_j \le 1 - e_{n_0}$ for all j, $e_{k_j} \perp p_j$ for $k_j > n_0$ and therefore

$$\left\|(e_{k_j} + p_j)\left(|u_{k_j}| + |u_{k_j}^*|\right)(e_{k_j} + p_j)\right\|^p \ge \left\|e_{k_j}\left(|u_{k_j}| + |u_{k_j}^*|\right)e_{k_j}\right\|^p + \left\|p_j\left(|u_{k_j}| + |u_{k_j}^*|\right)p_j\right\|^p$$

and taking the limit as $j \to \infty$, (4.1) and (4.2) imply $\alpha^p \ge \gamma^p + \alpha^p$. This is a contradiction since $\gamma > 0$. The proof of the lemma is complete.

To complete the proof of the theorem, assume that V is not uniformly integrable. Using Lemma 2.2 and Lemma 4.2,

$$\lim_{n\to\infty} \sup_{k\in\mathbb{N}} \|e_n v_k e_n\| > 0.$$

Again, choose a subsequence $(k_n) \subseteq \mathbb{N}$ so that

$$\lim_{n\to\infty} \|e_n v_{k_n} e_n\| > 0. \tag{4.3}$$

Claim: $\|e_n v_{k_n} e_n\|^2 \le 4\left\|(e_n - e_{k_n})\left(|u_{k_n}| + |u_{k_n}^*|\right)(e_n - e_{k_n})\right\|$.

To see this claim, we note that since $e_n \ge e_{k_n}$, $e_n v_{k_n} e_n = e_n u_{k_n} e_n - e_{k_n} u_{k_n} e_{k_n}$ so $e_n v_{k_n} e_n = (e_n - e_{k_n})u_{k_n} e_n + e_{k_n} u_{k_n}(e_n - e_{k_n})$ and therefore

$$\begin{aligned}
\|e_n v_{k_n} e_n\| &\le \|(e_n - e_{k_n})u_{k_n}\| + \|u_{k_n}(e_n - e_{k_n})\| \\
&\le \|u_{k_n}^*(e_n - e_{k_n})\| + \|u_{k_n}(e_n - e_{k_n})\| \\
&\le \|u_{k_n}^*\|^{\frac{1}{2}} \cdot \|(e_n - e_{k_n})|u_{k_n}^*|(e_n - e_{k_n})\|^{\frac{1}{2}} + \|u_{k_n}\|^{\frac{1}{2}} \cdot \|(e_n - e_{k_n})|u_{k_n}|(e_n - e_{k_n})\|^{\frac{1}{2}}
\end{aligned}$$

and since $\|u_{k_n}\| \le 1$,

$$\begin{aligned}
\|e_n v_{k_n} e_n\| &\le \|(e_n - e_{k_n})|u_{k_n}^*|(e_n - e_{k_n})\|^{\frac{1}{2}} + \|(e_n - e_{k_n})|u_{k_n}|(e_n - e_{k_n})\|^{\frac{1}{2}} \\
&\le 2\|(e_n - e_{k_n})\left(|u_{k_n}| + |u_{k_n}^*|\right)(e_n - e_{k_n})\|^{\frac{1}{2}}
\end{aligned}$$

and the claim follows.

From the claim above and equation (4.3), there exists $\nu > 0$ such that

$$\underline{\lim}_{\,n\to\infty} \left\|(e_n - e_{k_n})\left(|u_{k_n}| + |u_{k_n}^*|\right)(e_n - e_{k_n})\right\| = \nu > 0. \tag{4.4}$$

Observe that since $e_{k_n} \perp (e_n - e_{k_n})$,

$$\|e_n \left(|u_{k_n}| + |u^*_{k_n}|\right) e_n\|^p \geq \|e_{k_n} \left(|u_{k_n}| + |u^*_{k_n}|\right) e_{k_n}\|^p + \|(e_n - e_{k_n}) \left(|u_{k_n}| + |u^*_{k_n}|\right) (e_n - e_{k_n})\|^p.$$

Taking the limit (as $n \to \infty$) together with (4.1) and (4.4) would imply $\alpha^p \geq \nu^p + \alpha^p$. This is a contradiction since $\nu > 0$.

By setting $y_k := v_k$ and $z_k := e_k u_k e_k$, the proof for the σ-finite case is complete.

For the general case, let $\mathcal{M}$ be a von Neumann algebra (not necessarily σ-finite) and $(\varphi_n)_n$ in $L^p(\mathcal{M})$ as in the theorem. Fix an orthogonal family of cyclic projections $(e_\alpha)_{\alpha\in I}$ in $\mathcal{M}$ such that $1 = \vee_{\alpha\in I} e_\alpha$ (see for instance, [20, Proposition 5.5.9, p. 336]) .

Lemma 4.3. *There exists a countably decomposable projection $e \in \mathcal{M}$ such that for all $n \geq 1$, $e\varphi_n = \varphi_n e = \varphi_n$.*

For each $n \in \mathbb{N}$ and $\varepsilon > 0$, set $E_{n,\varepsilon} := \{\alpha \in I\ ;\ \|e_\alpha \varphi_n\| > \varepsilon\}$ and $E_n := \{\alpha \in I\ ;\ \|e_\alpha\varphi_n\| \neq 0\}$.

Claim: $E_{n,\varepsilon}$ is finite (hence E_n is countable).

To see this, assume that $E_{n,\varepsilon}$ is infinite. Then there exists an infinite sequence $(e_k)_k \subset (e_\alpha)_{\alpha\in I}$ such that $\|e_k\varphi_n\| > \varepsilon$ for all $k \in \mathbb{N}$. If J is a finite subset of $\mathbb{N}$, then

$$\begin{aligned}\|\sum_{k\in J} e_k\varphi_n\| &= \|(\sum_{k\in J} e_k)\varphi_n\|\\ &= \|(\vee_{k\in J} e_k)\varphi_n\| \leq \|\varphi_n\|.\end{aligned}$$

So $\|\sum_{k\in J} e_k\varphi_n\| \leq \|\varphi_n\|$ (a constant independant of J) which shows that $\sum_{k=1}^\infty e_k\varphi_n$ is a weakly unconditionally Cauchy (w.u.c.) series in $L^p(\mathcal{M})$ but since $L^p(\mathcal{M})$ does not contain any copies of c_0, $\sum_{k=1}^\infty e_k\varphi_n$ is unconditionally convergent and hence $\lim_{k\to\infty} \|e_k\varphi_n\| = 0$ (see for instance [5] p.45). This is in contradiction with the assumption $\|e_k\varphi_n\| \geq \varepsilon$ for all $k \in \mathbb{N}$. We proved that $E_{n,\varepsilon}$ is finite. It is clear that $E_n = \cup_{k\in\mathbb{N}} E_{n,\frac{1}{k}}$ so it is at most countable. The claim is verified.

Similarly, if $R_n = \{\alpha \in I, \|\varphi_n e_\alpha\| \neq 0\}$ then R_n is at most countable.

Let $C = \bigcup\limits_{n=1}^{\infty} (R_n \cup E_n)$; C is at most countable and if $e = \vee_{\alpha\in C} e_\alpha$ then e is the union of a countable family of disjoint cyclic projections in $\mathcal{M}$ so e is countably decomposable in $\mathcal{M}$ ([20, Proposition 5.5.19 p.340]). The construction of e implies that $e\varphi_n = \varphi_n e = \varphi_n$ for all $n \geq 1$. The lemma is proved.

To conclude the proof of the theorem, consider the von Neumann algebra $e\mathcal{M}e$. Since e is countably decomposable, $e\mathcal{M}e$ is σ-finite. Let $T : \mathcal{M} \to e\mathcal{M}e$ be the map that takes

$x \in \mathcal{M}$ to exe. The map T is bounded and is weak* to weak* continuous so there exists a map $S : (e\mathcal{M}e)_* \to \mathcal{M}_*$ so that $S^* = T$.

Let $R : \mathcal{M}_* \to (e\mathcal{M}e)_*$ be the restriction map. The operators T and R can be interpolated and since $L^p(\mathcal{M})$ (resp. $L^p(e\mathcal{M}e)$) is isometrically isomorphic to $(\mathcal{M}, \mathcal{M}_*)_\theta$ (resp. $(e\mathcal{M}e, (e\mathcal{M}e)_*)_\theta$) for $\theta = \frac{1}{p}$, (see [29]), we get a bounded linear map $T_p : L^p(\mathcal{M}) \to L^p(e\mathcal{M}e)$. Similarly, if one considers the inclusion map $e\mathcal{M}e \to \mathcal{M}$ and $S : (e\mathcal{M}e)_* \to \mathcal{M}_*$ as above, then by interpolation, we obtain a map $S_p : L^p(e\mathcal{M}e) \to L^p(\mathcal{M})$.

Apply the σ-finite case to the sequence $(T_p(\varphi_n))_{n\geq 1}$ in $L^p(e\mathcal{M}e)$ to get a decomposition

$$T_p(\varphi_{n_k}) = y_k + z_k \quad \forall\ k \geq 1$$

with $(y_k)_k$ and $(z_k)_k$ satisfying the conclusion of the theorem. It is enough to consider the decomposition:

$$\varphi_{n_k} = S_p(y_k) + S_p(z_k) \quad \forall\ k \geq 1.$$

The proof is complete. $\square$

The theorem which follows shows that, as in the semi-finite case, the decreasing projections in Theorem 4.1 can be replaced by mutually orthogonal projections. Its proof is identical to that of the semi-finite case ([26], Theorem 3.7).

Theorem 4.4. *Let $\mathcal{M}$ be a von Neumann algebra and $1 \leq p < \infty$, Let $(\varphi_n)_n$ be a bounded sequence in $L^p(\mathcal{M})$ then there exists a subsequence (φ_{n_k}) of φ_n, bounded sequences (y_k) and $(\zeta_k)_k$ in $L^p(\mathcal{M})$ and mutually orthogonal sequence of projections $(e_k)_k$ in $\mathcal{M}$ such that:*

(i) *$\varphi_{n_k} = y_k + \zeta_k$ for all $k \geq 1$;*
(ii) *$\{y_k : \ k \geq 1\}$ is uniformly integrable and $e_k y_k e_k = 0$ for all $k \geq 1$;*
(iii) *$(\zeta_k)_k$ is such that $e_k \zeta_k e_k = \zeta_k$ for all $k \geq 1$.*

Remark 4.5. *For $1 < p < \infty$, it should be noted that since $L^p(\mathcal{M})$ is is a closed subspace of $L^{p,\infty}(\mathcal{N}, \tau)$ and $L^{p,\infty}(\mathbb{R}^+, m)$ has the Fatou property, one could apply the semi-finite case of the the Kadec-Pełczyński subsequence decomposition to any bounded sequence of $L^p(\mathcal{M})$ (viewed as bounded sequence in $L^{p,\infty}(\mathcal{N}, \tau)$). However, that procedure would provide decreasing projections in $\mathcal{N}$ and as is noted in* [26, Remarks 3.5 (iii)], *these projections are either of finite trace or their orthogonal complements are of finite trace which guaranties that projections obtained from applying the semifinite case cannot be in $\mathcal{M}$.*

5. Applications

A result of Maurey ([23], see also [11]) states that every reflexive subspace of $L^1[0,1]$ has the fixed point property for nonexpansive mappings (FPP). Later, Dowling and Lennard

showed that the converse of Maurey's result is valid: every non-reflexive subspace $L^1[0,1]$ fails the FPP ([10]). This section is for the study of generalizations to the case of duals of C^*-algebras and requires the notion of asymptotically isometric copies of ℓ^1 which was introduced by Dowling and Lennard in [10].

Definition 5.1. *A Banach space X is said to contain asymptotically isometric copies of ℓ^1 if for every null sequence (ε_n) of positive numbers, there exists a sequence (x_n) in X such that:*

$$\sum_{n=1}^{\infty}(1-\varepsilon_n)|a_n| \leq \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq \sum_{n=1}^{\infty}|a_n|.$$

for all $(a_n) \in \ell^1$.

The following result is a generalization of [9].

Theorem 5.2. *let $\mathcal{A}$ be a C^*-algebra. Every non-reflexive subspace of $\mathcal{A}^*$ contains asymptotically isometric copies of ℓ^1.*

Proof. Note that $\mathcal{A}^{**}$ is a von Neumann algebra so subspaces of $\mathcal{A}^*$ are subspaces of preduals of von Neumann algebras. The proof then follows the argument used in [9] using Theorem 4.4. Details are left to the readers. □

Remark 5.3. *In* [3]*, Bélanger proved an improved version of the Akemann's characterization of weak compactness on preduals of von Neumann algebras. He then went on to show that non-reflexive preduals of von Neumann algebras contain complemented copies of ℓ^1. This fact can also be deduced from a result of Pfitzner* [25] *which states that C^*-algebras have Pełczyński property (V) so their duals have property (V*) (*[24]*). It is plain from Theorem 4.4 that the asymptotically isometric copies of ℓ^1 in Theorem 5.2 are complemented with good projection constants.*

For the next extension, we recall that JB^*-triples are all those Banach spaces whose open unit balls are bounded symmetric domains [30]. Examples of JB^*-triples are C^*-algebras and Hilbert spaces. Other important examples are the so-called *Cartan factors* $C^k(k=1,2,\dots,6)$ where the rectangular Cartan factor $C^1=\mathcal{L}(H,K)$ consists of bounded operators between Hilbert spaces, the symplectic factor C^2_n is $\{z \in \mathcal{L}(H);\ z=-jz^*j\}$ where $j: H \to H$ is a conjugate linear isometric involution, the Hermitian Cartan factor C^3 is $\{z \in \mathcal{L}(H);\ z=jz^*j\}$, C^4 is the spin factor, C^5 is the (finite dimentional) exceptional Cartan factor consisting of 1×2 matrices over the complex Caley numbers $\mathbf{O}$ and C^6 is the set of all 3×3 Hermitian matrices over $\mathbf{O}$. Dual JB^*-triples are called JBW^*-triples. For more informations, we refer to [4], [15] and [16].

Corollary 5.4. *If $\mathcal{J}$ is a JB^*-triple then every non-reflexive subspace of $\mathcal{J}^*$ contains asymptotically isometric copies of ℓ^1.*

For the poof we will need two lemmas on stability of asymptotically isometric copies of ℓ^1.

Lemma 5.5. *Let E_1 and E_2 be weakly sequentially complete Banach spaces so that any sequence equivalent to the unit vector basis of ℓ^1 in E_j $(j = 1, 2)$ has a normalized block that is asymptotically isometric to ℓ^1 then every sequence equivalent to the unit vector basis of ℓ^1 in $E_1 \oplus_1 E_2$ has a normalized block that is asymptotically isometric to ℓ^1.*

Proof. Let $\{U_n = (x_n, y_n)\}_{n=1}^{\infty}$ be a sequence in $E_1 \oplus_1 E_2$ that is equivalent to ℓ^1. After taking subsequences, either $(x_n)_n$ or $(y_n)_n$ is equivalent to ℓ^1. Let assume that $(x_n)_n$ is equivalent to ℓ^1. We have two cases.
Case 1: The sequence $(y_n)_n$ is weakly convergent. By taking normalized blocks, we can assume that $(x_n)_n$ is asymptotically isometric to ℓ^1 and $\lim_{n\to\infty} \|y_n\| = 0$. There exists a null sequence (ε_n) of positive numbers such that:

$$\sum_{n=1}^{\infty}(1-\varepsilon_n)|a_n| \leq \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq \sum_{n=1}^{\infty} |a_n|.$$

for all $(a_n) \in \ell^1$ but since $\| \sum_{n=1}^{\infty} a_n U_n\| = \| \sum_{n=1}^{\infty} a_n x_n\| + \| \sum_{n=1}^{\infty} a_n y_n\|$, we get that

$$\sum_{n=1}^{\infty}(1-\varepsilon_n)|a_n| \leq \left\|\sum_{n=1}^{\infty} a_n U_n\right\| \leq \sum_{n=1}^{\infty}(1+\|y_n\|)|a_n|.$$

This concludes that $(U_n)_n$ is asymptotically isometric to ℓ^1.
Case 2: The sequence (y_n) is equivalent to ℓ^1. As above, one can find a block so that both the coresponding block for $(x_n)_n$ and $(y_n)_n$ are asymptotically isometric to ℓ^1. Set $Z_n := U_n/2 = (x_n/2, y_n/2)$. It can be easily seen that $(Z_n)_n$ is equivalent to an asympotically isometric copy of ℓ^1 in $E_1 \oplus_1 E_2$. □

Lemma 5.6. *Let $(\Omega, \Sigma, \lambda)$ be a measure space and R be a reflexive Banach space. Every sequence equivalent to the unit vector basis of ℓ^1 in $L^1(\lambda, R)$ has a normalized block that is asymptotically isometric to ℓ^1.*

Proof. Let $(f_n)_n$ be a sequence equivalent to the ℓ^1 basis. Since R is reflexive, the sequence $(f_n)_n$ can not be uniformly integrable (see for instance [6]). Apply the classical Kadec-Pełczyński subsequence decomposition to the sequence $(\|f_n(\cdot)\|)_n$ in $L^1(\lambda)$ to get a pairwise disjoint sequence of measurable sets $(A_n)_n$ such that $\left\{\|f_n(\cdot)\|\chi_{\Omega\setminus A_n}, n \geq 1\right\}$ is uniformly integrable. The space R being reflexive implies that $\left\{f_n\chi_{\Omega\setminus A_n}, n \geq 1\right\}$ is relatively weakly compact in $L^1(\lambda, R)$. We conclude the proof as in the scalar case. □

Proof of Corollary 5.4: Let $\mathcal{J}$ be a JB^*-triple and X be a non-reflexive subspace of $\mathcal{J}^*$. Since $\mathcal{J}^{**}$ is a JBW^*-triple, we can assume that X is a subspace of the predual of a JBW^*-triple $\mathcal{I}$. By [15] and [16], $\mathcal{I}$ admits the following form:

$$\mathcal{I} = \left(\sum_{\alpha} \oplus C(\Omega_\alpha, C^\alpha) \right)_{\ell^\infty} \oplus_\infty J^7 \oplus_\infty J^8,$$

where $C(\Omega_\alpha, C^\alpha)$ is the space of continuous functions from a hyperstonean space Ω_α to a Cartan factor C^α, $J^7 = \{a \in M; \Theta(a) = a\}$ with $\Theta : M \to M$ is a w^*-continuous * -antiautomorphism of period 2 on a von Neumann algebra M and J^8 is a w^*-closed right ideal of a von Neumann algebra N. The predual of $\mathcal{I}$ is equal to the ℓ^1-sum

$$\mathcal{I}_* = \left(\sum_{\alpha} \oplus L^1(\Sigma_\alpha, C^\alpha_*) \right)_{\ell^1} \oplus_1 J^7_* \oplus_1 J^8_*.$$

By [4, Theorem 2], the space $E_1 = \left(\sum_{\alpha \neq 5,6} \oplus L^1(\Sigma_\alpha, C^\alpha_*) \right)_{\ell^1} \oplus_1 J^7_* \oplus_1 J^8_*$ is isometric to a 1-complemented subspace of the predual of a von Neumann algebra so E_1 is isometric to a subspace of the predual of such von Neumann algebra and hence satisfies the assumption of Lemma 5.3. Moreover, since C^5 and C^6 are finite dimensional, the space $E_2 = L^1(\Sigma_5, C^5) \oplus_1 L^1(\Sigma_6, C^6)$ satisfies (as does L^1-spaces) the assumption of Lemma 5.3. We conclude that every sequence equivalent to the unit vector basis of ℓ^1 in $\mathcal{I}_* = E_1 \oplus_1 E_2$ has a normalized block that is asymptotically isometric to ℓ^1. The proof is complete. □

Corollary 5.7. *If $\mathcal{J}$ is a JB^*-triple then every non-reflexive subspace of $\mathcal{J}^*$ fails the fixed point property for nonexpansive self-maps on closed bounded convex sets.*

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